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MATRICES

MATRIX

An arrangement of certain numbers in an array of m rows and n columns is called as m X n matrix (read as m by n). Such as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots a_{2n} \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots a_{mn} \end{bmatrix}$$

In the above arrangement, all horizontal lines correspond to rows and vertical lines correspond to columns.

m denotes total number of rows in A, n denotes total number of columns in A. thus order of A is written as $A_{m \times n.}$ in short, matrix A can also be denoted by

$A = \{a_{ij}\}_{m \times n} where\ a_{ij}$ Denotes an element belonging to the i^{th} row and j^{th} column where I = 1, 2, 3… m; j = 1, 2 ….n.

$$A_{3\times2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}, \qquad A_{3\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

Different notations used for enclosing the elements of matrix are [ ], ( ), II ll, { }.

It must be noted that matrix represents the arrangement of numbers and it satisfies certain rules and laws of operations. It cannot be evaluated like a determinant. While determinant represents certain number or is a notation to denote certain expression, matrix is an entity in the form of arrangement of numbers, which cannot be disturbed.

Elements a_{11}, a_{22}, a_{33}, ….etc. are said to be on leading diagonal or principal diagonal and they are termed as diagonal elements.

1.1 TYPES OF MATRICES

1. Square matrix: A matrix containing number of rows = number of columns is known as square matrix.

$$A_{3\times3} = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 3 & 1 \\ 1 & 6 & 2 \end{bmatrix}, \qquad A_{2\times2} = \begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix}$$

2. Diagonal matrix: If a square matrix contains all non-diagonal elements as zeroes then it is called as diagonal matrix.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

3. Row or column matrices: A matrix of order 1 X n i. e. having only one row and n columns is known as row matrix.

$$e.g. A = [\, a_{11}\; a_{12}\; a_{13} \;\ldots\ldots a_{in}], \qquad B_{14} = [\,5, 7, 0, 4]$$

A matrix of order m x 1 i.e. having only one column and m rows, is known as column matrix.

$$\text{e.g. } A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \ldots \\ a_{m1} \end{bmatrix}, B_{31} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

4. Zero or null matrix: A matrix containing all zero elements is known as zero or null matrix and it is denoted by Z.

$$\text{e. g. } Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Transpose of matrix: matrix obtained by interchanging rows and columns of a matrix A is called transpose of A. it is denoted by A' or A* or A^T.

$$If\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} then\ A^T = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 0 \\ 3 & 0 & 2 & 1 \end{bmatrix}$$

Properties:

1) $(A+B)^T = \mathrm{A^T} + \mathrm{B^T}$

2) $(A^T)^T = \mathrm{A}$

3) $(AB)^T = \mathrm{A^T B^T}$

6. symmetric matrix: A square matrix A is said to be symmetric matrix if A = A'

i.e. A = $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 5 & 3 & 2 \end{bmatrix}$ also A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

In a symmetric matrix, $a_{ij} = a_{ji}\ for\ all\ i, j.$

$$a_{12} = a_{21}, a_{13} = a_{31}, a_{23} = a_{32}$$

7. Skew symmetric matrix: A is said to be skew symmetric if $A = -A^T$

Note: Diagonal elements of skew symmetric matrix are zero

8. Scalar matrix: If all diagonal elements of square matrix are equal then it is scalar matrix.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

9. Unit/identity matrix: It is a diagonal matrix having all diagonal elements is one.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. Lower triangular matrix: It is a square matrix having all elements above diagonal is zeroes

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 5 & 1 & 8 \end{bmatrix}$$

11. Upper triangular matrix: It is a square matrix having all elements below diagonal is zeroes

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 0 & 9 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Trace of a Matrix

Let A is a square matrix of order n. The sum of the elements lying along principle diagonal is called the trace of A denoted by Tr(A)

$$Tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + a_{33} + \ \dots \dots . a_{nn}$$

$$A = \begin{bmatrix} 3 & 4 & 3 \\ 7 & 3 & 0 \\ 5 & 1 & -2 \end{bmatrix}$$

$$Tr(A) = 3 + 3 - 2 = 4$$

Properties of Trace of a matrix

Consider A and B are two square matrices of order n and λ be a scalar

1) $Tr(\lambda A) = \lambda \, Tr(A)$

2) $Tr(A + B) = Tr(A) + Tr(B)$

3) $Tr(AB) = Tr(BA)$

Determinant of Matrix

If A is square matrix then $|A|$ = determinant of A.

$$\text{If } A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\text{Then } |A| = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{vmatrix}$$

Expansion: $|A| = 4\,(0 - 3) - \ 3\,(\,0 + 3) + \ 2\,(\,1 + 2) = \ -12 - 9 + 6 = \ -15$

Minor of an element of $|A|$

$$\text{Let } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then minor of an element of $|A|$ is a determinant obtained by omitting the row and the column in which the element is present.

$$minor\ of\ a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \quad minor\ of\ a_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$minor\ of\ b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \quad minor\ of\ c_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

In general, minor of an element a_{ij} is denoted by M_{ij}.

COFACTOR OF AN ELEMENT a_{ij} in $|A|$

Cofactor of an element a_{ij} $in\ |A|$ $is\ denoted\ by\ A_{ij}$. it is a minor of a_{ij} associated with proper sign.

$$\text{e.g. } A_{ij} = (-1)^{i+j}\ M_{ij}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{21} = (-1)^{2+1} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

ADJOINT OF A MATRIX

Adjoint of a square matrix A is the transpose of the matrix by the cofactors of the elements of the given matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

e. g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 4 & 5 & -4 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -1 & 5 \\ 3 & -3 & -4 \end{bmatrix}$$

C. F. of 1 = 19 C. F. of 0 = 23 C. F. of 4 = -3

C. F. of 2 = -12 C. F. of -1 = -16 C. F. of 5 = 3

C. F. of 3 = 4 C. F. of -3 = 3 C. F. of -4 = -1

$$\text{Adj A} = \begin{bmatrix} 19 & 23 & -3 \\ -12 & -16 & 3 \\ 4 & 3 & -1 \end{bmatrix}$$

Inverse of A

If A is a square matrix and $|A| \neq 0$ i.e. A is a non – singular matrix then

$$A^{-1} = \frac{1}{|A|}\, adj\ A.$$

1.2 OPERATIONS OF MATRICES

1. Equality of two matrices: if two matrices A and B have same order and all the elements of A and B in the corresponding positions are equal then A = B.

$$e.g. A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\ A = B$$

2. Multiplication of matrix by scalar:

$$if\ A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \end{bmatrix} then\ 2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & -2 & 8 \end{bmatrix}$$

3. Sum of matrices: if A and B have order then we define A + B as a matrix obtained by addition of corresponding elements of A an B.

$$i.e. A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\ A + B = \begin{bmatrix} 1+2 & 2+3 \\ 1+4 & 1+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 6 \end{bmatrix}$$

Multiplication of matrices: If column of A = rows of B then product AB exits

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\times 4 + 2\times 2 & 1\times 5 + 2\times 3 \\ 3\times 4 + 4\times 2 & 3\times 5 + 4\times 3 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 20 & 28 \end{bmatrix}$$

Properties:

1) $A^2 = AA$

2) $AB \neq BA$

3) $If\ AB = AC\ does\ not\ imply\ B = C$

1.3 Rank of a matrix

The matrix is said to be of rank r if there is

1. At least one minor of the order r which is not equal to zero and

2. Every minor of the order (r + 1) is equal to zero.

The rank of matrix A is the maximum order of its non- vanishing minor and it is denoted as

$$\rho(A) = r$$

If a matrix has a non – zero minor of order r, then $\rho(A) \geq r$

If a matrix has all the minors of order (r + 1) as zeroes, then $\rho(A) \leq r$

If A is an m x n matrix then P(A) ≤ minimum of m and n.

Elementary transformations of a matrix do not alter the rank and order of a matrix.

TYPE I : FINDING THE RANK BY REDUCING THE MATRIX A TO ECHELON FORM

Echelon or canonical form of A matrix: let a matrix A be of order m X n i.e. $A_{m \times n.}$

Then the canonical or echelon form A is a matrix in which

1. One or more elements in each of the first r rows are non-zero and the elements in the remaining rows are zeroes.

Echelon form of matrix $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Ex. 1: find the rank of A, where $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

Sol : $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

Perform $R_2 - 3R_1$; $R_3 + R_1 \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$

Perform $R_3 + R_2 \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Here m = total no. of rows = 3

K = total no. of rows which contain all zero elements = 1.

ρ (A) = m – k = 3 – 1 = 2, P (A) = 2

Ex. 2: find the rank of $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Sol: $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Perform $R_{12} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Perform $R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1 \sim$ of $A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Perform R_2 - $R_3 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Perform $R_3 - 4R_2, R_4 - 9R_2 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$

Perform $\frac{1}{11}R_3, \frac{1}{22}R_4 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix}$

Perform R_4 - $R_3 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Here m = 4, k = 1

ρ(A) = m – k = 4 – 1 = 3 hence ρ (A) = 3.

TYPE II: FINDING THE RANK BY REDUCING THE MATRIX A TO THE NORMAL FORM

Definition: by performing elementary transformations, any non-zero matrix A can be reduced to one of the following four forms, called the normal form of A.

(1) [I_r]

(2) [I_r 0]

(3) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

(4) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Where I_r is unit matrix or identity matrix and 0 denotes null matrix.

Ex. 1: Reduce the following matrix to its normal form and hence find its rank, where

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Sol :

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

By R_{12}
$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$by\ R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$
$$\sim \begin{bmatrix} 1 & -1 & -2 & 4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$By\ C_2 + C_1, C_3 + 2C_1, C_4 +\ 4C_1$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$BY\ R_2 - R_3$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$BY\ R_3 - 4R_2, R_4 - 9R_2$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$BY\ C_3 +\ 6C_2, C_4 + 3C_2$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$BY\ \frac{1}{33}C_3, \frac{1}{22}C_4$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$BY\ R_4 - 2R_3$ $$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$BY\ C_4 - C_3$ $$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

A = $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ which is the required normal form of A. hence P(A) = 3.

1.4 Inverse of matrix

Adjoint method

If (1) A is square matrix.

(2) $|A| \neq 0$ Then there exists A^{-1} and it is given by

$$\boxed{A^{-1} = \frac{1}{|A|} adj. A.}$$

Ex. 1: find A^{-1} by using adjoint method for A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Sol: $|A| = 1\,(18 - 12) - 1\,(9 - 3) + 1\,(4 - 2) = 6 - 6 + 2 = 2$

$$A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

C.F. of 1 = 6   C.E. of 1 = -5 C.F. of 1 = 1

C.F. of 1 = -6   C.E. of 2 = 8 C.F. of 4 = -2

C.F. of 1 = 2   C.E. of 3 = -3 C.F. of 9 = 1

Adj A = $\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

Ex. 2: find A^{-1} by using adjoint method A = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Sol: $|A| = 1\ (3+0) - 2\ (-1-0) - 2\ (2-0) = 1$

$$A' = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

C.F. of 1 = 3 C.F. of -1 = 2 C.F. of 0 = 6

C.F. of 2 = 1 C.F. of 3 = 1 C.F. of -2 = 2

C.F. of -2 = 2 C.F. of 0 = 2 C.F. of 1 = 5

$$\text{Adj. } A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Solution of simultaneous equation by matrices

Consider a system of ‘m’ linear equations in ‘n’ unknowns $X_1, X_2, X_3 \ldots X_n$

$$a_{11}\ x_1 + a_{12}\ x_2 + \cdots + a_{in}\ x_n = b_1$$

$$a_{21}\ x_1 + a_{22}\ x_2 + \cdots + a_{2n}\ x_n = b_2$$

$$a_{m1}\ x_1 + a_{m2}\ x_2 + \cdots + a_{mn}\ x_n = b_m$$

The above system of equations can be written in compact form by using metrix notation

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

i.e. AX = B, where coefficient matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

When the system AX = B has a solution i.e. set of values of $x_1, x_2, \dots x_n$ satisfy simultaneously all m equations then system is said to be consistent otherwise the system is called inconsistent.

AUGMENTED MATRIX (A, B)

Definition: if AX = B is a system of m equations in n unknowns then matrix written as (A, B) is called as the augmented matrix.

Hence, (A, B) = $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \vdots & \vdots & \dots & \vdots & : & \vdots \\ \vdots & \vdots & \dots & \vdots & : & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$

NON – HOMOGENEOUS EQUATIONS

For the system of equations AX = B if matrix B is not a null or zero matrix then the system AX = B is known as non – homogeneous system of equations.

HOMOGENEOUS EQUATIONS

For the system of equations AX = B if matrix B is a null or zero matrix i.e. Z then the system AX = Z is known as homogeneous system equations.

e.g. (1) $\left.\begin{matrix} x+y+z=3 \\ x-y+2z=4 \\ 2x+3y-z=0 \end{matrix}\right\}$ (2) $\left.\begin{matrix} x+y+z=0 \\ 2x-3y+4z=0 \\ x-y+2z=0 \end{matrix}\right\}$

Non – homogeneous equations homogenous equations

1.5 CONDITION FOR CONSISTENCY OF NON-HOMOGENEOUS EQUATIONS

Consider a system of 'm' equations in 'n' unknowns given by AX = B.

Let m = total number of equations, n = total number of unknowns.

Case 1: let m $\neq$ n (i.e. no. of equation different from number of unknowns).

(1) If rank of the coefficient matrix A = rank of the augmented matrix (A, B), then the system AX = B is said to be consistent. In other words, if ρ (A) = ρ (A, B) then system is consistent.

(2) No solution: if ρ (A) $\neq$ ρ (A, B) then system is said to be inconsistent and possesses no solution.

(3) A unique solution: if $\rho(A) = \rho(A, B) = n =$ total number of unknowns, then system AX = B possesses a unique solution (only one solution).

(4) An infinite number of solutions: if $\rho(A) = \rho(A, B) < n$ then system possesses an infinite number of solutions.

Case 2: let m = n. no. of equations = no. of unknowns > 3.

(1) $\rho(A) = \rho\ (A, B) = n\ system\ is\ consistent\ and\ possesses\ a\ unique\ solution.$

$(2)\rho(A) \neq \rho\ (A, B) =$ System is inconsistent and possesses no solution.

(3) $\rho(A) = \rho\ (A, B) =$ r < n then system is consistent and possesses an infinite number of solutions.

Case 3: let m = n. No. of equations = no. of unknowns = 3. i.e. 3 equations in 3 unknowns. We have AX = B. find $|A|$

(1) If $|A| \neq 0$ system is consistent, then A^{-1} exists and system possesses a unique solution given by X = A^{-1} B.

(2) If $|A| = 0$ then system is either inconsistent or possesses an infinite number of solutions and this is decided by applying the method of reduction. Write augmented matrix (A, B).

Now, if $\rho(A) = \rho(A, B) < n$ then system possesses an infinite number of solutions

If $\rho(A) \neq \rho(A, B)$ then system is inconsistent and possesses no solution

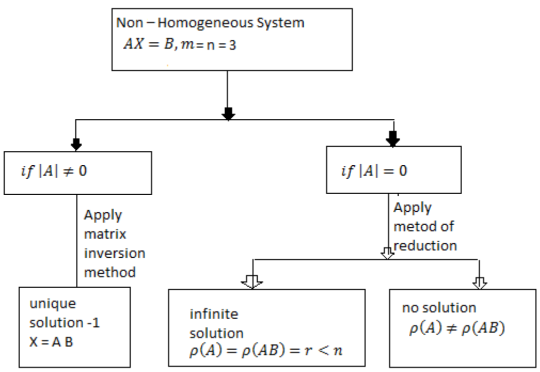

chart 2

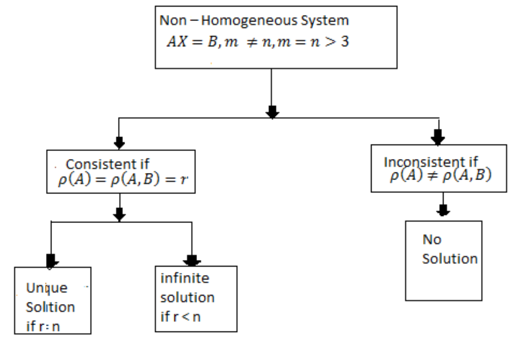

Chart 1

Ex. 1: solve the system of equations by matrix method

$$2x_1 + x_2 - x_3 + 3x_4 = 8$$

$$x_1 + x_2 + x_3 - x_4 + 2 = 0$$

$$3x_1 + 2x_2 - x_3 = 6$$

$$4x_2 + 3x_3 + 2x_4 + 8 = 0$$

Sol: given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 0 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \\ 6 \\ -8 \end{bmatrix}$$

AX = B.

Consider an augmented matrix.

$$(A, B) = \begin{bmatrix} 2 & 1 & -1 & 3 & | & 8 \\ 1 & 1 & 1 & -1 & | & -2 \\ 3 & 2 & -1 & 0 & | & 6 \\ 0 & 4 & 3 & 2 & | & -8 \end{bmatrix}$$

Perform R_{12}

$$\begin{bmatrix} 1 & 1 & 1 & -1 & | & -2 \\ 2 & 1 & -1 & 3 & | & 8 \\ 3 & 2 & -1 & 0 & | & 6 \\ 0 & 4 & 3 & 2 & | & -8 \end{bmatrix}$$

$$\text{Perform } R_2 - 2R_1,\ R_3 - 3R_1 \sim \begin{bmatrix} 1 & 1 & 1 & -1 & | & -2 \\ 0 & -1 & -3 & 5 & | & 12 \\ 0 & -1 & -4 & 3 & | & 12 \\ 0 & 4 & 3 & 2 & | & -8 \end{bmatrix}$$

$$\text{Perform } R_3 - R_2,\ R_4 + 4R_2 \sim \begin{bmatrix} 1 & 1 & 1 & -1 & | & -2 \\ 0 & -1 & -3 & 5 & | & 12 \\ 0 & 0 & -1 & -2 & | & 0 \\ 0 & 0 & -9 & 22 & | & 40 \end{bmatrix}$$

$$\text{Perform } R_4 - 9R_3 \sim \begin{bmatrix} 1 & 1 & 1 & -1 & | & -2 \\ 0 & -1 & -3 & 5 & | & 12 \\ 0 & 0 & -1 & -2 & | & 0 \\ 0 & 0 & 0 & 40 & | & 40 \end{bmatrix}$$

$\rho(A) = \rho(A, B) = 4 - 0 = 4 =$ total number of variables

Hence system possesses a unique solution and it is given as follows.

$By\ R_4$ $\quad 40\ x_4 = 40,$ $\quad x_4 = 1$

$By\ R_3$ $\quad -x_3 - 2x_4 = 0,$ $\quad x_3 = -2$

$By\ R_2$ $\quad -x_2 - 3x_3 + 5x_4 = 12,$ $\quad x_2 = -3(-2) + 5(1) - 12 = -1$

$By\ R_1$ $\quad x_1 + x_2 + x_3 - x_4 = -2,$ $\quad x_1 = -2 + 1 + 2 + 1 = 2$

Hence the solution set is $x_1 = 2, \quad x_2 = -1, \quad x_3 = -2, \quad x_4 = 1$

Ex. 2: Find the value of k for an infinite number of solutions.

2x – 3y + 6z – 5t = 3

Y – 4z + t = 1

4x – 5y + 8z – 9t = k

A) 4 B) 5 C) 6 D) 7

Sol: step 1: given system in matrix form can be written as,

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}$$

Step 2 : consider an augmented matrix,

$$(A, B) = \begin{bmatrix} 2 & -3 & 6 & -5 & | & 3 \\ 0 & 1 & -4 & 1 & | & 1 \\ 4 & -5 & 8 & -9 & | & k \end{bmatrix}$$

Step 3: now we will reduce (A, B) to echelon form.

Perform $R_3 - 2R_1$ $\sim= \begin{bmatrix} 2 & -3 & 6 & -5 & | & 3 \\ 0 & 1 & -4 & 1 & | & 1 \\ 0 & 1 & -4 & 1 & | & k-6 \end{bmatrix}$

Perform $R_3 - R_2$ $\sim \begin{bmatrix} 2 & -3 & 6 & -5 & | & 3 \\ 0 & 1 & -4 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & | & k-7 \end{bmatrix}$

For an infinite number of solutions

$$\rho(A) = \rho(A, B) = r < n$$

$\rho(A) = 3 - 1 = 2$

$$\rho(A) = \rho(A, B) = 2$$

$$k - 7 = 0$$

$$k = 7$$